Bott periodicity

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Stable homology through scanning
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Goal: understand topology of $GL_n\mathbb{C}$.

First reduction: Gram-Schmidt gives a deformation retraction of $GL_n\mathbb{C}$ onto $U(n)$, the compact group of $n\times n$ unitary matrices (preserving the inner product on $\mathbb{C}^n$).

Since $U(n+1)$ preserves length, it acts on the unit sphere $S^{2n+1}\subset\mathbb{C}^{n+1}$; the stabilizer of a vector is $U(n)$, giving a fiber bundle $U(n)\to U(n+1)\to S^{2n+1}$

$\Rightarrow \pi_i(U(n))\approx\pi_i(U(n+1))$ for $i<2n$.

Examples:

$\pi_0(U(n))=0$

$U(1)=S^1$, so $\pi_1(U(1))=\mathbb{Z}$ $\Rightarrow \pi_1(U(n))=\mathbb{Z}$, detected by determinant: $U(n)\to S^1$

$S^1\to U(2)\to S^3$, so $\pi_2(U(2))=0$ $\Rightarrow \pi_2(U(n))=0$ (true for any Lie group)

$\pi_3(U(2))=\mathbb{Z}$ $\Rightarrow \pi_3(U(n))=\mathbb{Z}$ we see a pattern emerging...

Bott periodicity: $\pi_i(U(n))\approx\pi_{i+2}(U(n))$ for $n\gg i$.

Let $U=\lim_{n\to\infty}U(n)=\bigcup_{n\geq 1}U(n)$. Fixing $H=\mathbb{C}^\infty$, U = unitary transformations of H which are the identity on a co-finite dimensional subspace of H

Bott periodicity $\Leftrightarrow \pi_i(U)\approx\pi_{i+2}(U)$ for all i $\Leftrightarrow U\simeq\Omega^2U$

Since we know $\Omega BU\simeq U$ for any group, it suffices to prove that $BU\simeq\Omega U$

raising π_i same as lowering π_i

Of course this is false: BU is connected, while $\pi_0(\Omega U)=\pi_1(U)=\mathbb{Z}$. But this is easily fixed.

Bott periodicity, rephrased: $\mathbb{Z}\times BU\simeq\Omega U$.

Outline of proof:

Let $K=\coprod_{n\geq 1}BU(n)$. K classifies complex vector bundles, so direct sum of vector bundles corresponds to an operation $K\times K\to K$ which we describe later.

We will show that $BK\simeq U$.

Then the group completion theorem gives $H_*(K)[\pi_0^{-1}]\approx H_*(\Omega U)$.

But $\pi_0(K)=\mathbb{N}$, so inverting π_0 group-completes π_0 to $\mathbb{Z}$, and stabilizes $H_*(BU(n))$ to $H_*(BU)$:

$$H_*(K)[\pi_0^{-1}]=H_*\Big(\coprod_{n\geq 1}BU(n)\Big)[\pi_0^{-1}]=H_*(\mathbb{Z}\times BU),$$

and both $\mathbb{Z}\times BU$ and ΩU are simply connected
($\pi_1(BU)=\pi_0(U)=0$) ($\pi_1(\Omega U)=\pi_2(U)=0$)

so by the Hurewicz + Whitehead theorems,

a homology equivalence $H_*(\mathbb{Z}\times BU)\approx H_*(\Omega U)$

is a homotopy equivalence $\mathbb{Z}\times BU\simeq\Omega U$, as desired.

Understanding $K = \coprod_{n \geq 1} BU(n)$

Let Fr_n = space of orthonormal n-frames in H.
$U(n)$ acts on Fr_n with quotient
Gr_n = space of n-planes in H.
Since Fr_n is contractible, we have $Gr_n = Fr_n / U(n) = BU(n)$
Thus $K = \coprod_{n \geq 1} BU(n)$ is the space of finite-dimensional subspaces of H.

What is our operation $K \times K \to K$? it suffices to define it on each component $BU(n) \times BU(m) \to \coprod_{k \geq 0} BU(k)$.

We take the map $BU(n) \times BU(m) \to BU(n+m)$
induced by the inclusion $U(n) \times U(m) \hookrightarrow U(n+m)$ $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$

[Note for later:
Let $Gr_{n,m}$ = space of pairs (V_1, V_2) with V_1 an n-plane in H, and V_2 an m-plane in H, with V_1 and V_2 orthogonal.
Then $Gr_{n,m} = Fr_{n+m} / U(n) \times U(m)$, so $Gr_{n,m} = B(U(n) \times U(m))$
and the map $Gr_{n,m} \to Gr_{n+m}$, $(V_1, V_2) \mapsto V_1 \oplus V_2$ realizes the map $B(U(n) \times U(m)) \to BU(n+m)$ induced by inclusion $U(n) \times U(m) \hookrightarrow U(n+m)$]

The relaxation principle suggests that we take as a model for BK the space of all $(\underset{V_1}{|}\ \underset{V_2}{|} \cdots \underset{V_n}{|})$ finite subsets of $(0,1)$ labeled by subspaces V_i of H topologized to vanish at the boundary

each element is determined by $n \geq 0$ subspaces $V_1, \ldots, V_n \in K$
and by the separations $(\overset{t_0}{\ }|\overset{t_1}{\ }| \cdots |\overset{t_n}{\ })$ satisfying $t_i > 0$, $t_0 + \cdots + t_n = 1$
So as a <u>set</u> this space is $BK = \coprod_{n \geq 0} \underset{V_1}{\underset{\uparrow}{K}} \times \cdots \times \underset{V_n}{\underset{\uparrow}{K}} \times \underset{(t_0, \ldots, t_n)}{\underset{\uparrow}{\Delta^n}}$

but we need to topologize so that
as $t_1 \to 0$ and V_1 and V_2 come together, we replace them by $V_1 * V_2 \in K$.

Problem: to do this explicitly, we need a concrete realization of the map $*: Gr_n \times Gr_m \to Gr_{n+m}$, which takes some work.
Plus, even given this, it takes a lot of bookkeeping to make it associative (as V_1, V_2, and V_3 come together simultaneously, what do we replace them with?).

Solution: restrict to work with subspace on which direct sum has a nice geometric model. ③

Let Y = space of all  finite subsets of $(0,1)$ labeled by $V_i < H$ s.t. V_i and V_j are orthogonal, topologized so that as V_i and V_{i+1} come together ($t_i \to 0$), we replace them by their span $V_i \oplus V_{i+1}$, and V_1 and V_n can disappear at the boundary ($t_0 \to 0$, $t_n \to 0$).

Claim: $Y \simeq BK$.

The space of n-simplices in Y is the space of sequences $V_1, \dots, V_n$ with $V_i < H$ and V_i orthogonal to V_j.

Ex: for a 2-point set we have an n-plane V_1 and an m-plane V_2 that are orthogonal. The space of such pairs is $Gr_{n,m} = B(U(n) \times U(m))$; note that the projections $Gr_n \leftarrow Gr_{n,m} \rightarrow Gr_m$ induce a homotopy equivalence $Gr_{n,m} \simeq Gr_n \times Gr_m$.

Thus $Y = \coprod_{n \geq 0} \coprod_{k_1, \dots, k_n} Gr_{k_1, \dots, k_n} \times \Delta^n / \sim$

while $BK = \coprod_{n \geq 0} K \times \cdots \times K \times \Delta^n / \sim$

$= \coprod_{n \geq 0} \coprod_{k_1, \dots, k_n} Gr_{k_1} \times \cdots \times Gr_{k_n} \times \Delta^n / \sim$.

Since $Gr_{k_1, \dots, k_n} \simeq Gr_{k_1} \times \cdots \times Gr_{k_n}$, the spaces of n-simplices are homotopy equivalent, and this implies that $Y \simeq BK$.

It remains to show that $Y \simeq U$.

Instead of points in $\longleftrightarrow$, think of Y as points in .

So a point in Y is a finite collection of complex numbers $\lambda_i \neq 1$ in the unit circle, and for each λ_i a finite-dimensional subspace $V_i < H$ which are orthogonal.

But the spectral theorem states that every unitary matrix is unitarily diagonalizable with eigenvalues on the unit circle and eigenspaces orthogonal.

As two eigenvalues come together, the corresponding eigenspaces are combined; as an eigenvalue goes to 1, we need no longer keep track of its eigenspace.

Conclusion:

Our model $Y = BK = B(\coprod BU(n))$ is not only homotopy equivalent to U, it is exactly <u>homeomorphic</u> to U.